

2. Waveform data processing

- Fourier transform
- Time domain, amplitude & phase, real & imaginary
- Ambient noise

Almost all processing of active/passive surface wave methods will be done in frequency domain.

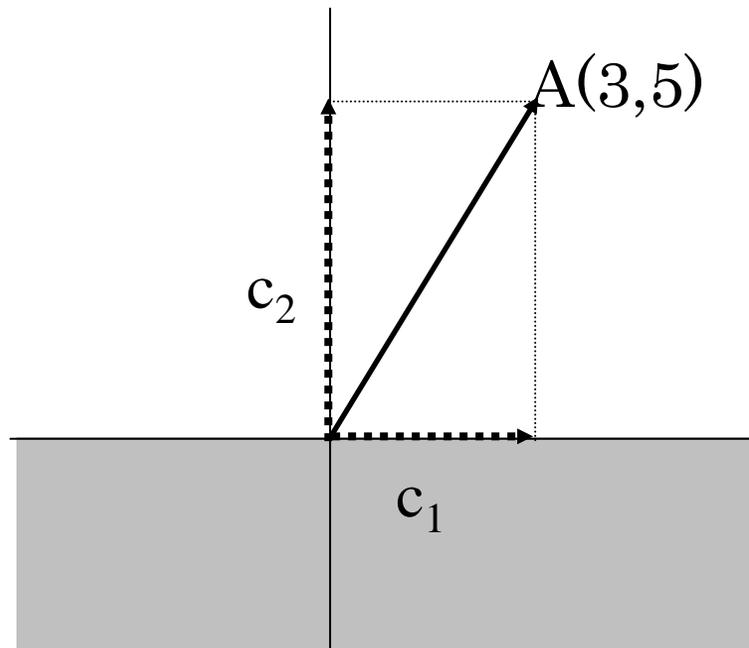
Understanding frequency domain and Fourier transform is quite important.

Fourier transform is

1. Orthogonal
The inner product of any two different harmonics is zero
2. Complete
No function can be orthogonal to all harmonics
3. Convenient
Can be calculate efficiency

Vector decomposition

Vector decomposition

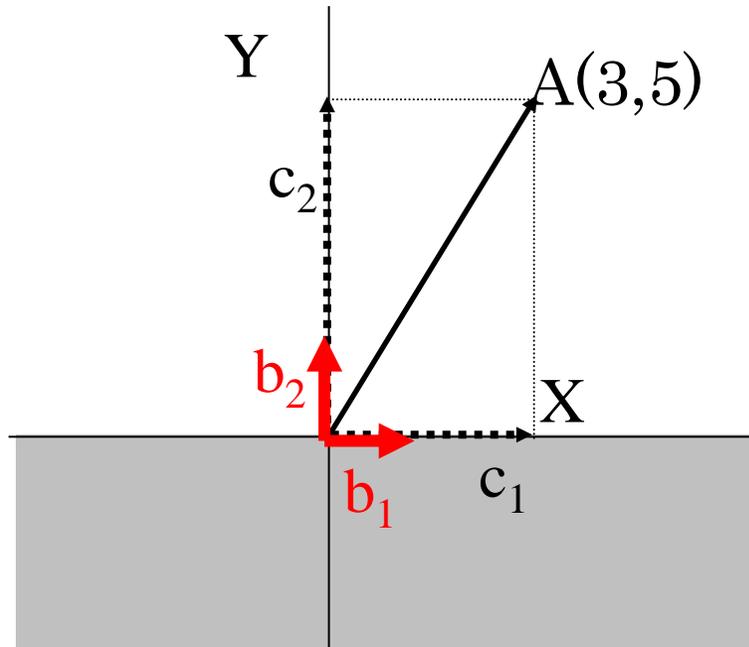


$$A = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$$



Let's Decompose into C_1 and C_2 !

Vector decomposition



Unit vectors

$$X = b_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad Y = b_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Calculating inner product !

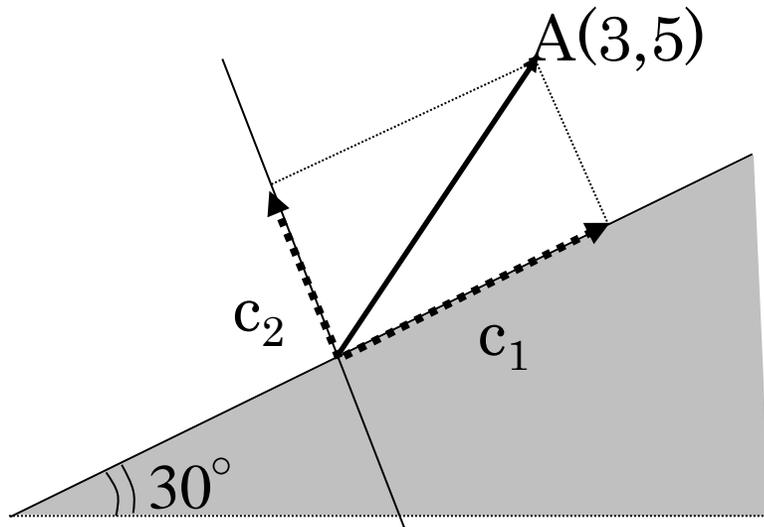
$$c_1 = b_1^T A = (1 \quad 0) \begin{pmatrix} 3 \\ 5 \end{pmatrix} = 3$$

$$c_2 = b_2^T A = (0 \quad 1) \begin{pmatrix} 3 \\ 5 \end{pmatrix} = 5$$

$$\downarrow$$
$$C = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = B^T A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$$

Vector decomposition

Vector decomposition with slope



Unit vectors

$$b_1 = \begin{pmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \quad b_2 = \begin{pmatrix} -\frac{1}{2} \\ \frac{2}{\sqrt{3}} \\ \frac{2}{2} \end{pmatrix}$$

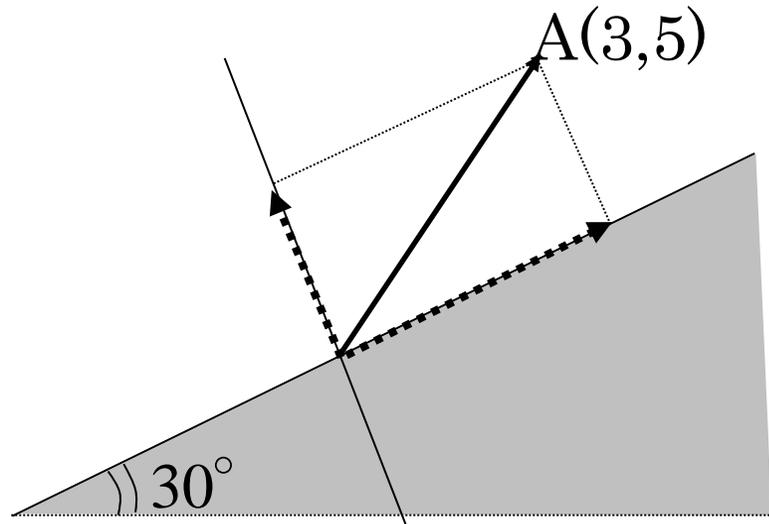
Remember! Inner product is 0

$$b_1 \cdot b_2 = \frac{\sqrt{3}}{2} \times \frac{-1}{2} + \frac{1}{2} \times \frac{\sqrt{3}}{2} = 0$$

$$C = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = B^T A = \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} \begin{pmatrix} 3 \\ 5 \end{pmatrix} = \begin{pmatrix} \frac{3\sqrt{3} + 5}{2} \\ \frac{5\sqrt{3} - 3}{2} \end{pmatrix} = \begin{pmatrix} 5.098 \\ 2.830 \end{pmatrix}$$

Vector decomposition

Vector decomposition with slope



Inverse transform

B and C \longrightarrow A

$$A = (B^T)^{-1} C = BC = \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{-1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{1}{2} & \frac{2}{2} \end{pmatrix} \begin{pmatrix} \frac{3\sqrt{3}+5}{2} \\ \frac{5\sqrt{3}-3}{2} \end{pmatrix} = \begin{pmatrix} \frac{9+5\sqrt{3}-5\sqrt{3}+3}{4} \\ \frac{3\sqrt{3}+5+15-3\sqrt{3}}{4} \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$$

Vector decomposition and orthogonal transform

Orthogonal transform

$$C = B^T A$$

$$C = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = B^T A = \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} \begin{pmatrix} 3 \\ 5 \end{pmatrix} = \begin{pmatrix} \frac{3\sqrt{3}+5}{2} \\ \frac{5\sqrt{3}-3}{2} \end{pmatrix} = \begin{pmatrix} 5.098 \\ 2.830 \end{pmatrix}$$

Inverse transform

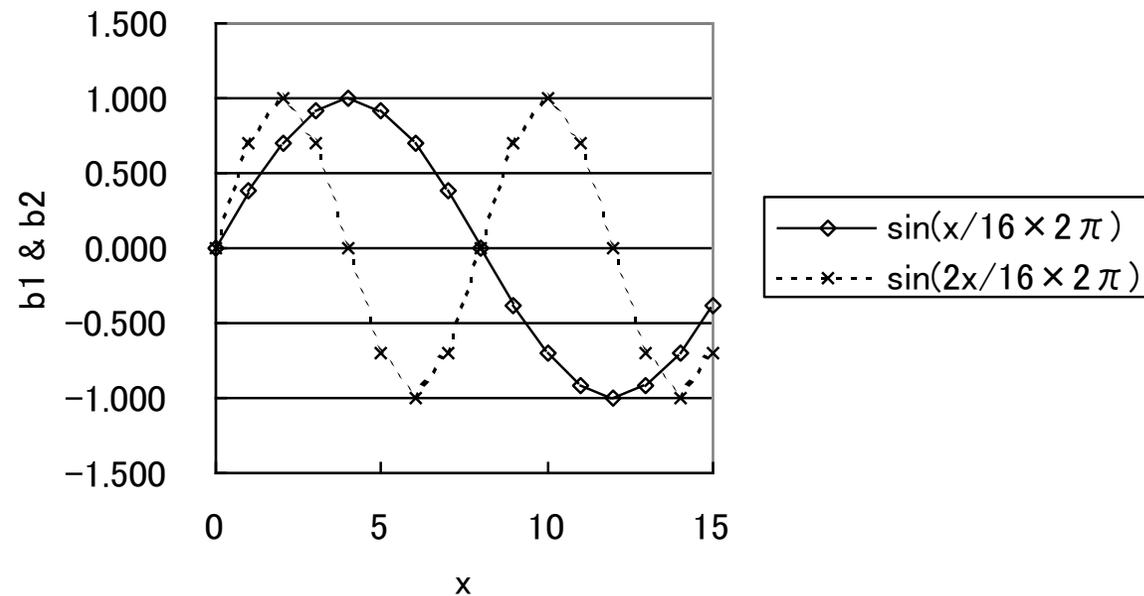
$$A = BC$$

$$A = (B^T)^{-1} C = BC = \begin{pmatrix} \frac{\sqrt{3}}{2} & -1 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} \begin{pmatrix} \frac{3\sqrt{3}+5}{2} \\ \frac{5\sqrt{3}-3}{2} \end{pmatrix} = \begin{pmatrix} \frac{9+5\sqrt{3}-5\sqrt{3}+3}{4} \\ \frac{3\sqrt{3}+5+15-3\sqrt{3}}{4} \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$$

Trigonometric functions are orthogonal

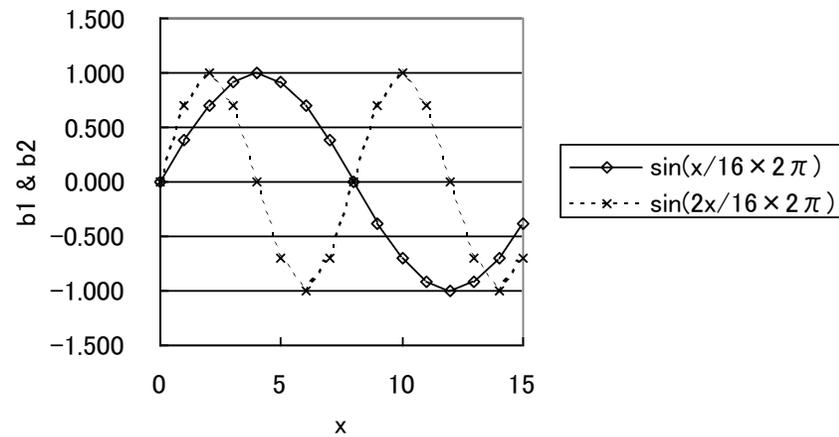
Two trigonometric functions (Two vectors)

$$b_1 = \sin\left(1 \times \frac{x}{16} \times 2\pi\right) \quad b_2 = \sin\left(2 \times \frac{x}{16} \times 2\pi\right)$$



Trigonometric functions are orthogonal

Inner product

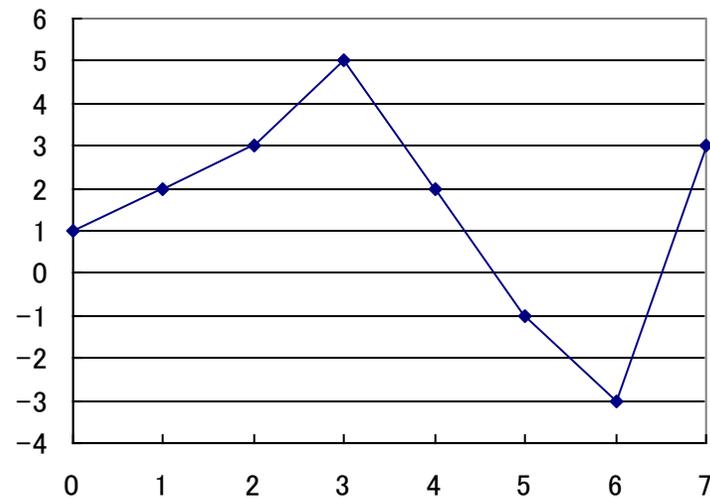


x	$b_1 = \sin(1 \times x/16 \times 2\pi)$	$b_2 = \sin(2 \times x/16 \times 2\pi)$	$b_1 \times b_2$
0	0.000	0.000	0.000
1	0.383	0.707	0.271
2	0.707	1.000	0.707
3	0.924	0.707	0.653
4	1.000	0.000	0.000
5	0.924	-0.707	-0.653
6	0.707	-1.000	-0.707
7	0.383	-0.707	-0.271
8	0.000	0.000	0.000
9	-0.383	0.707	-0.271
10	-0.707	1.000	-0.707
11	-0.924	0.707	-0.653
12	-1.000	0.000	0.000
13	-0.924	-0.707	0.653
14	-0.707	-1.000	0.707
15	-0.383	-0.707	0.271
		Total	0.000

Discrete Fourier transform

Let's transform 8 samples discrete waveform data!

Waveform



Vector notation



$$A = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 5 \\ 2 \\ -1 \\ -3 \\ 3 \end{pmatrix}$$

Discrete Fourier transform

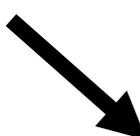
Matrix consists of 8 trigonometric functions

$$B_{ij} = \begin{pmatrix} \sin\left(0 \times \frac{0}{8} \times 2\pi\right) & \sin\left(1 \times \frac{0}{8} \times 2\pi\right) & \cdot & \cdot & \cdot & \cdot & \cdot & \sin\left(7 \times \frac{0}{8} \times 2\pi\right) \\ \sin\left(0 \times \frac{1}{8} \times 2\pi\right) & \cdot & & & & & & \cdot \\ \sin\left(0 \times \frac{2}{8} \times 2\pi\right) & & \cdot & & & & & \cdot \\ \cdot & & & \cdot & & & & \cdot \\ \cdot & & & & \cdot & & & \cdot \\ \cdot & & & & & \cdot & & \cdot \\ \sin\left(0 \times \frac{7}{8} \times 2\pi\right) & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \sin\left(7 \times \frac{7}{8} \times 2\pi\right) \end{pmatrix} = \left(\sin\left(n \times \frac{x}{8} \times 2\pi\right) \right)$$

Discrete Fourier transform

$$B_{ij} = \begin{pmatrix} \sin\left(0 \times \frac{0}{8} \times 2\pi\right) & \sin\left(1 \times \frac{0}{8} \times 2\pi\right) & \cdot & \cdot & \cdot & \cdot & \cdot & \sin\left(7 \times \frac{0}{8} \times 2\pi\right) \\ \sin\left(0 \times \frac{1}{8} \times 2\pi\right) & \cdot & & & & & & \cdot \\ \sin\left(0 \times \frac{2}{8} \times 2\pi\right) & & \cdot & & & & & \cdot \\ \cdot & & & \cdot & & & & \cdot \\ \cdot & & & & \cdot & & & \cdot \\ \cdot & & & & & \cdot & & \cdot \\ \cdot & & & & & & \cdot & \cdot \\ \sin\left(0 \times \frac{7}{8} \times 2\pi\right) & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \sin\left(7 \times \frac{7}{8} \times 2\pi\right) \end{pmatrix} = \left(\sin\left(n \times \frac{x}{8} \times 2\pi\right) \right)$$

Extract $n \times x$



$$W_{ij} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 0 & 2 & 4 & 6 & 8 & 10 & 12 & 14 \\ 0 & 3 & 6 & 9 & 12 & 15 & 18 & 21 \\ 0 & 4 & 8 & 12 & 16 & 20 & 24 & 28 \\ 0 & 5 & 10 & 15 & 20 & 25 & 30 & 35 \\ 0 & 6 & 12 & 18 & 24 & 30 & 36 & 42 \\ 0 & 7 & 14 & 21 & 28 & 35 & 42 & 49 \end{pmatrix} = (n \times x)$$

Discrete Fourier transform

Real part (cos)

$$B_{\text{Re}} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0.71 & 0 & -0.71 & -1 & -0.71 & 0 & 0.71 \\ 1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 \\ 1 & -0.71 & 0 & 0.71 & -1 & 0.71 & 0 & -0.71 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -0.71 & 0 & 0.71 & -1 & 0.71 & 0 & -0.71 \\ 1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 \\ 1 & 0.71 & 0 & -0.71 & -1 & -0.71 & 0 & 0.71 \end{pmatrix} = \cos\left(\frac{1}{8} \times 2\pi \times W_{ij}\right)$$

Imaginary part (sin)

$$B_{\text{Im}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.71 & -1 & -0.71 & 0 & 0.71 & 1 & 0.71 \\ 0 & -1 & 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & -0.71 & 1 & -0.71 & 0 & 0.71 & -1 & 0.71 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.71 & -1 & 0.71 & 0 & -0.71 & 1 & -0.71 \\ 0 & 1 & 0 & -1 & 0 & 1 & 0 & -1 \\ 0 & 0.71 & 1 & 0.71 & 0 & -0.71 & -1 & -0.71 \end{pmatrix} = -\sin\left(\frac{1}{8} \times 2\pi \times W_{ij}\right)$$

Discrete Fourier transform

Fourier transform is

$$A = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 5 \\ 2 \\ -1 \\ -3 \\ 3 \end{pmatrix}$$

$$C_{\text{Re}} = B_{\text{Re}}^T A$$



$$C_{\text{Im}} = B_{\text{Im}}^T A$$

$$C_{\text{Re}} = \begin{pmatrix} 12.00 \\ -0.29 \\ 3.00 \\ -1.71 \\ -6.00 \\ -1.71 \\ 3.00 \\ -0.29 \end{pmatrix}$$

$$C_{\text{Im}} = \begin{pmatrix} 0.00 \\ -9.54 \\ 7.00 \\ 2.46 \\ 0.00 \\ -2.46 \\ -7.00 \\ 9.54 \end{pmatrix}$$

Discrete Fourier transform

Inverse Fourier transform is

$$C_{\text{Re}} = \begin{pmatrix} 12.00 \\ -0.29 \\ 3.00 \\ -1.71 \\ -6.00 \\ -1.71 \\ 3.00 \\ -0.29 \end{pmatrix}$$

$$A_{\text{Re}} = \frac{1}{8} (B_{\text{Re}} C_{\text{Re}} - (-B_{\text{Im}}) C_{\text{Im}})$$

$$A_{\text{Re}} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 5 \\ 2 \\ -1 \\ -3 \\ 3 \end{pmatrix}$$



$$C_{\text{Im}} = \begin{pmatrix} 0.00 \\ -9.54 \\ 7.00 \\ 2.46 \\ 0.00 \\ -2.46 \\ -7.00 \\ 9.54 \end{pmatrix}$$

$$A_{\text{Im}} = \frac{1}{8} (B_{\text{Re}} C_{\text{Im}} + (-B_{\text{Im}}) C_{\text{Re}})$$

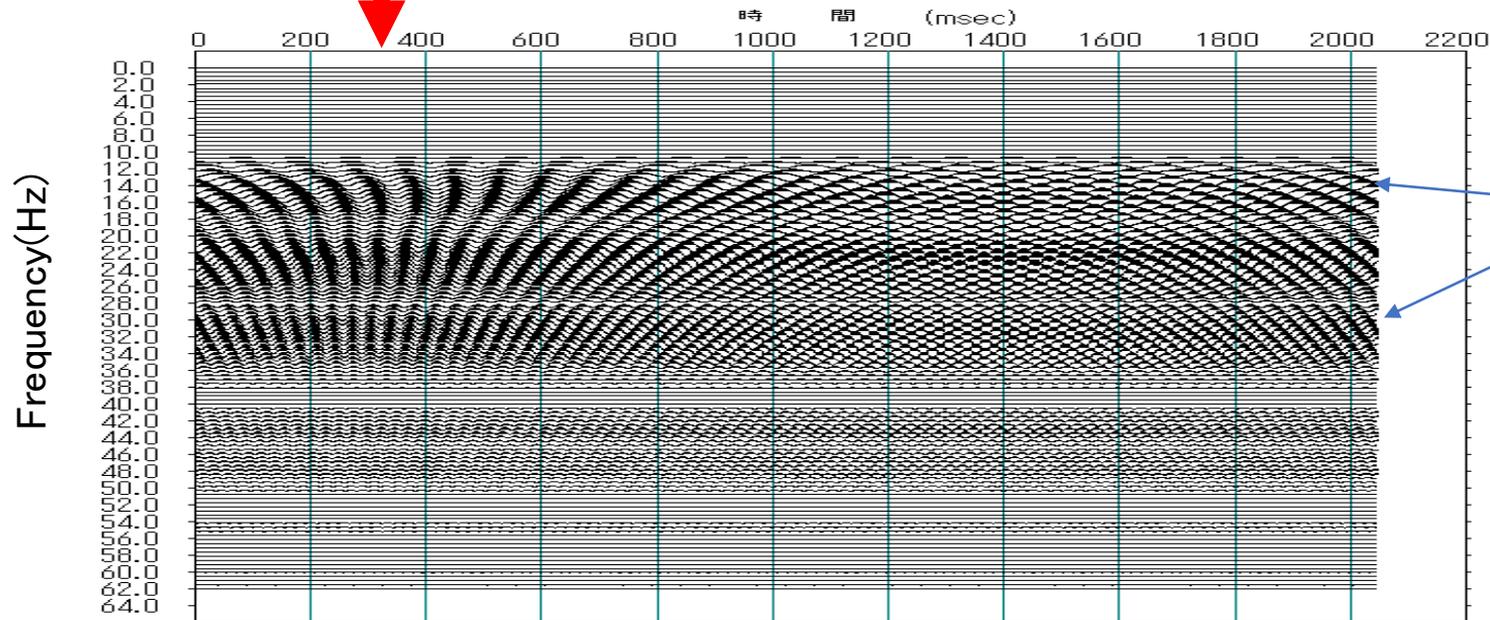
Fourier series

Any waveform data can be decomposed into trigonometric functions!

$$f(t) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{2\pi n t}{T} + b_n \sin \frac{2\pi n t}{T} \right)$$



Decompose into each frequencies



Trigonometric functions

Fourier transform

Calculating Fourier coefficients (a_0, a_1 to a_k, b_1 to b_k)

$$f(t) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{2\pi n t}{T} + b_n \sin \frac{2\pi n t}{T} \right)$$

Fourier transform

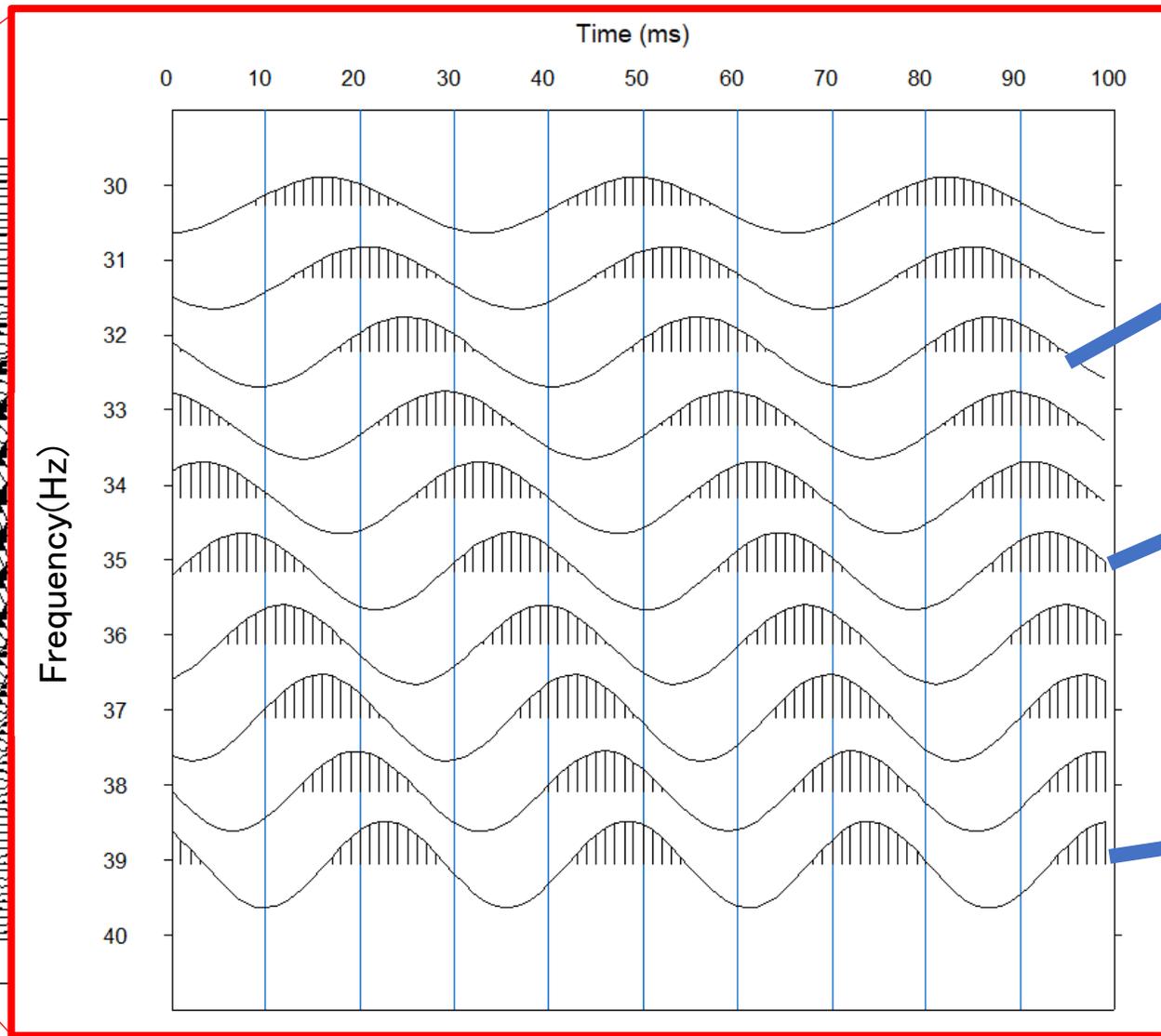
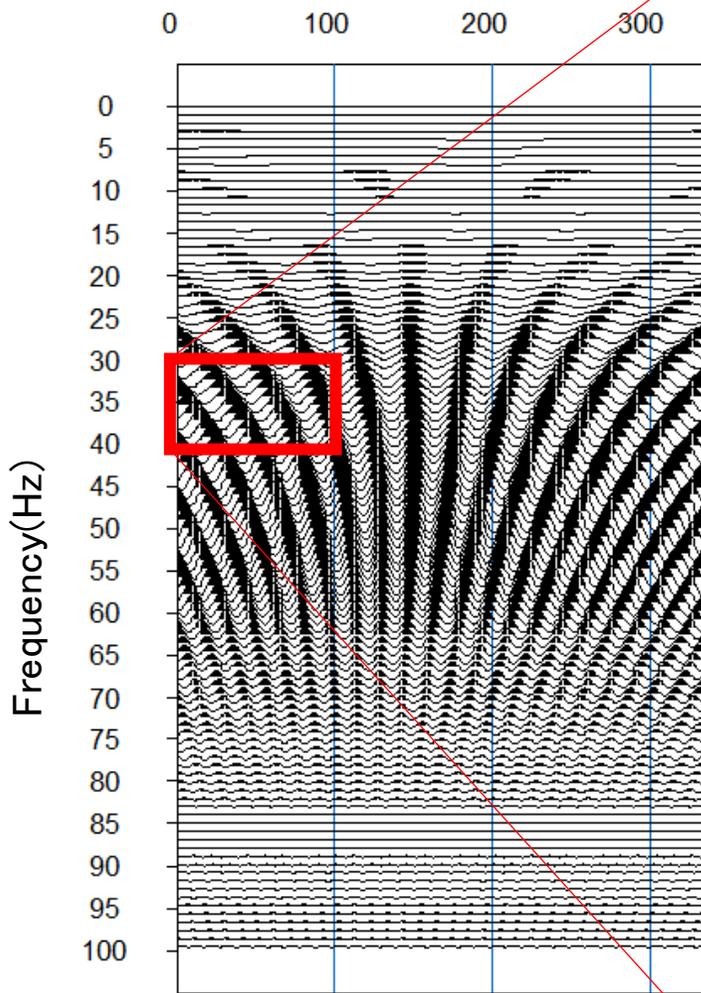
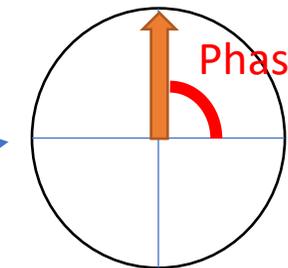
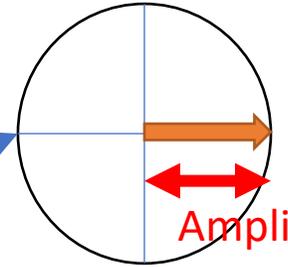
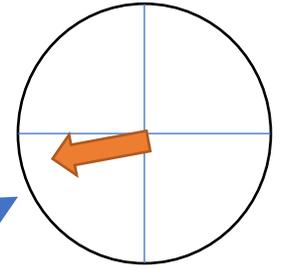
$$F(\omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(t) \cdot \exp^{-i\omega t} dt$$

Inverse Fourier transform

$$f(t) = \int_{-\infty}^{+\infty} F(\omega) \cdot \exp^{-i\omega t} d\omega$$

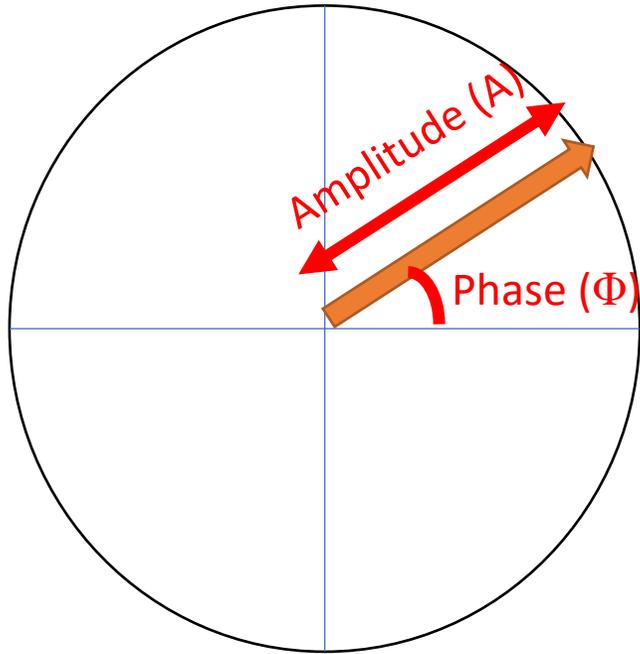
Amplitude and phase in frequency domain

Where trigonometric function starts?

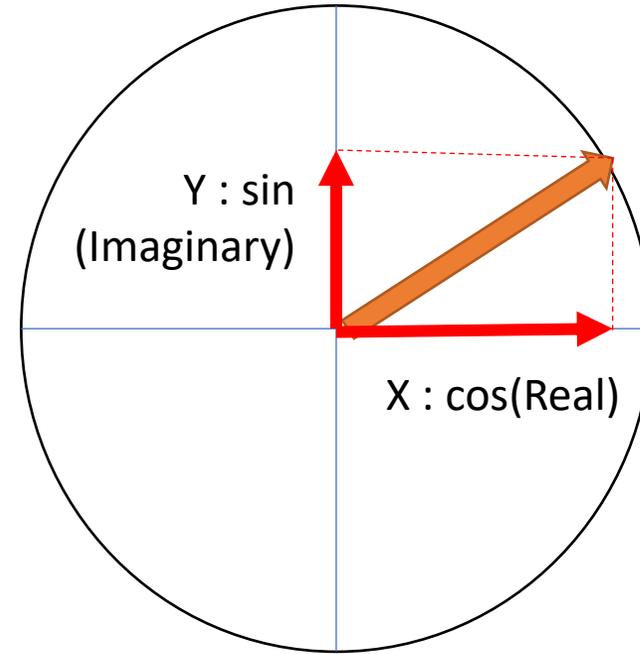


Amplitude and phase in terms of complex number

Complex number
with amplitude (A) and phase (Φ)

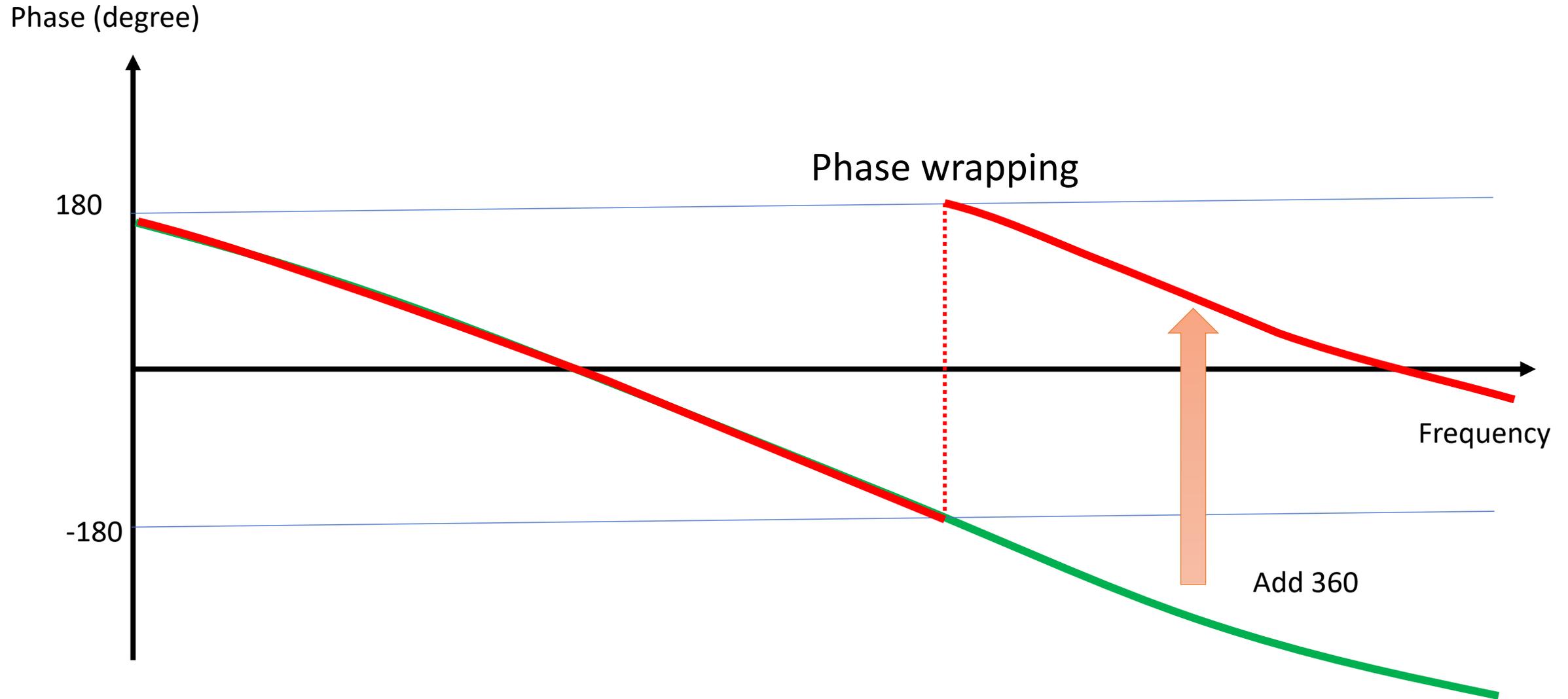


Complex number
with cos (X : Real) and sin (Y : Imaginary)



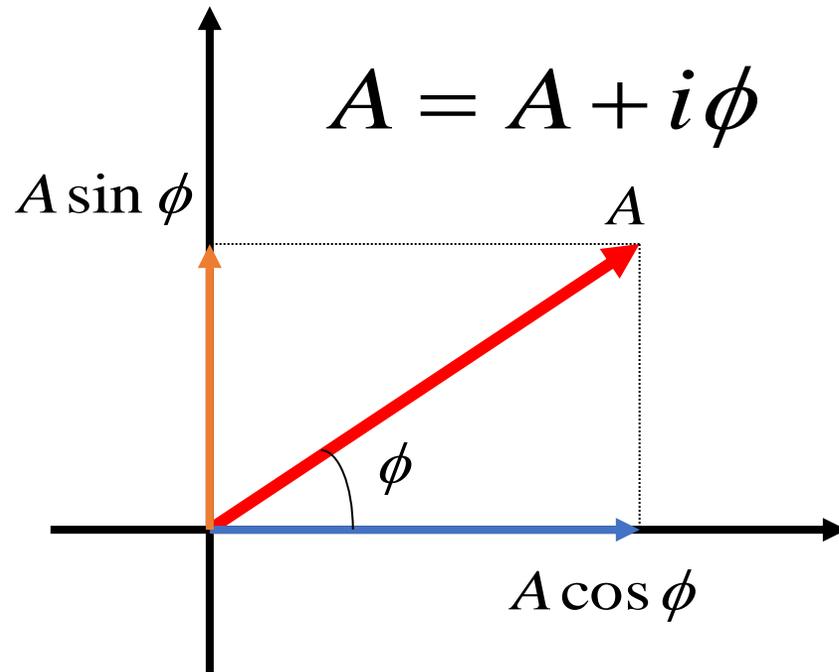
Phase is defined between $0 \sim 360$ ($-180 \sim 180$) degrees
or $0 \sim 2\pi$ ($-\pi$ to π)

Phase is defined between $0 \sim 360$ ($-180 \sim 180$) degrees



Complex numbers

Complex plane with amplitude (A) and phase (Φ)

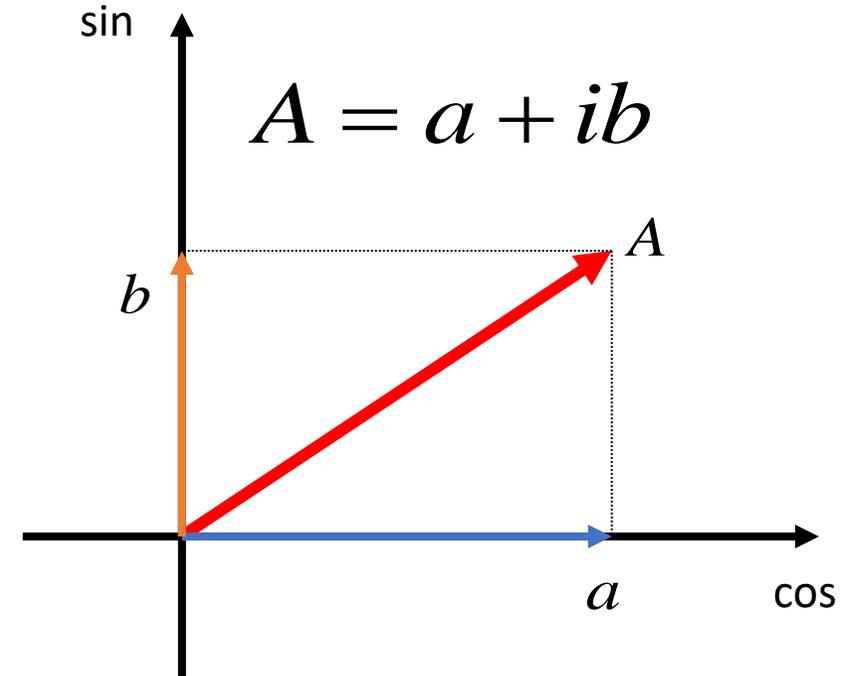


$$A = A + i\phi$$

$$A = A \cos \phi + iA \sin \phi$$

$$\phi = \tan^{-1} \frac{b}{a}$$

Complex plane with cos and sin



$$A = a + ib$$

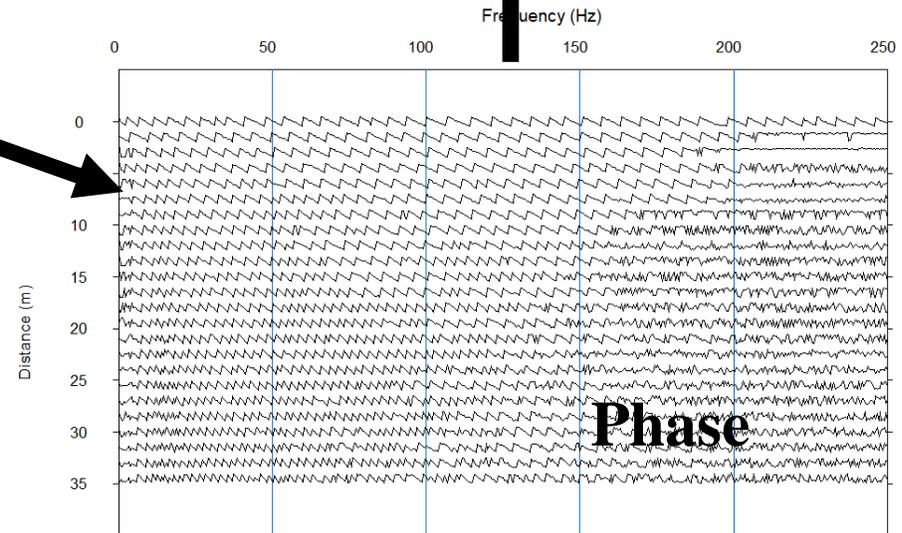
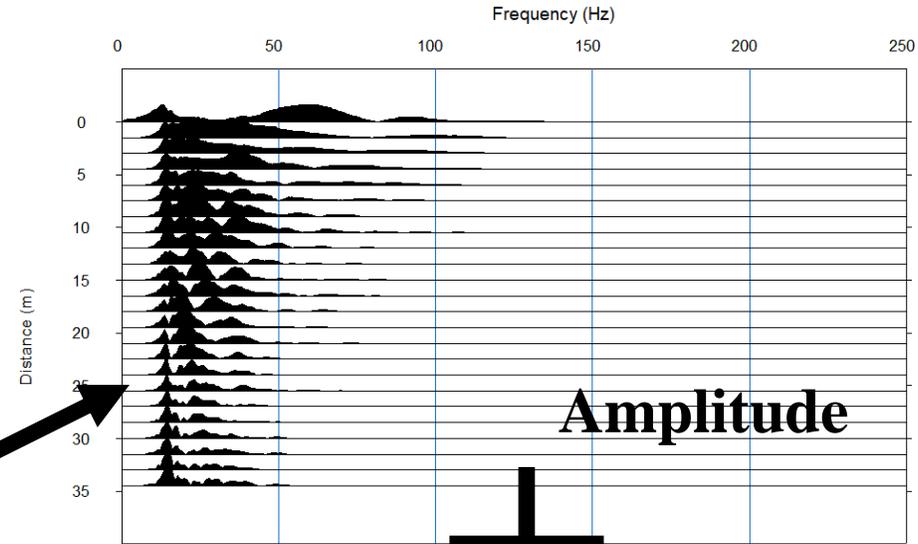
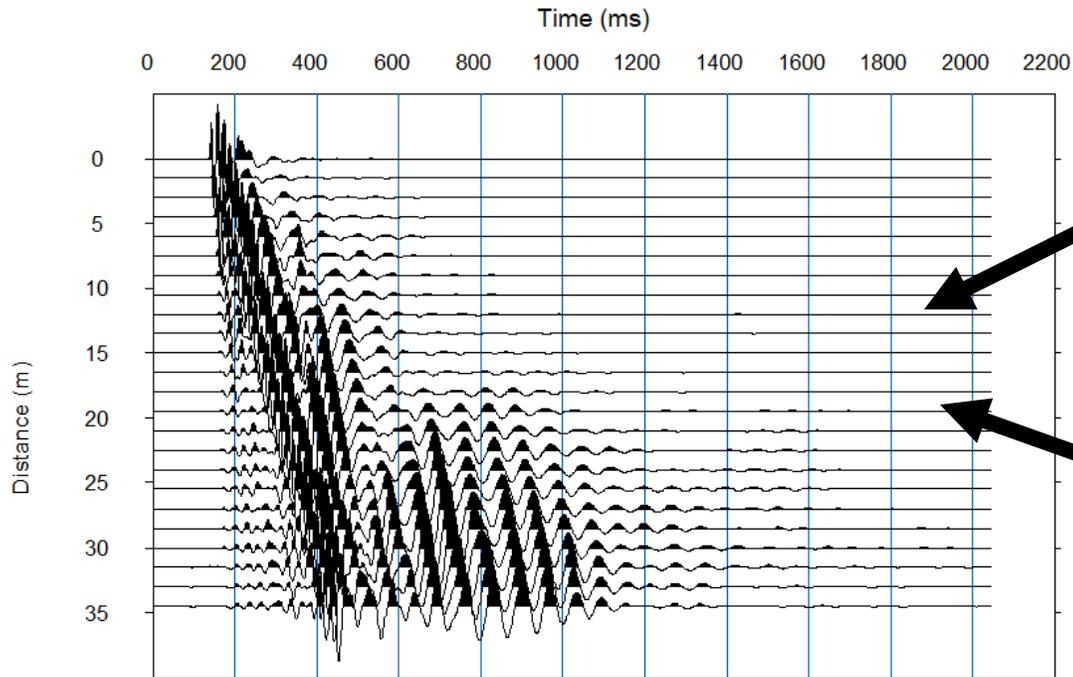
$$A = a \cos \frac{2\pi t}{T} + b \sin \frac{2\pi t}{T}$$

$$A = \sqrt{a^2 + b^2}$$

Amplitude and phase

Frequency domain

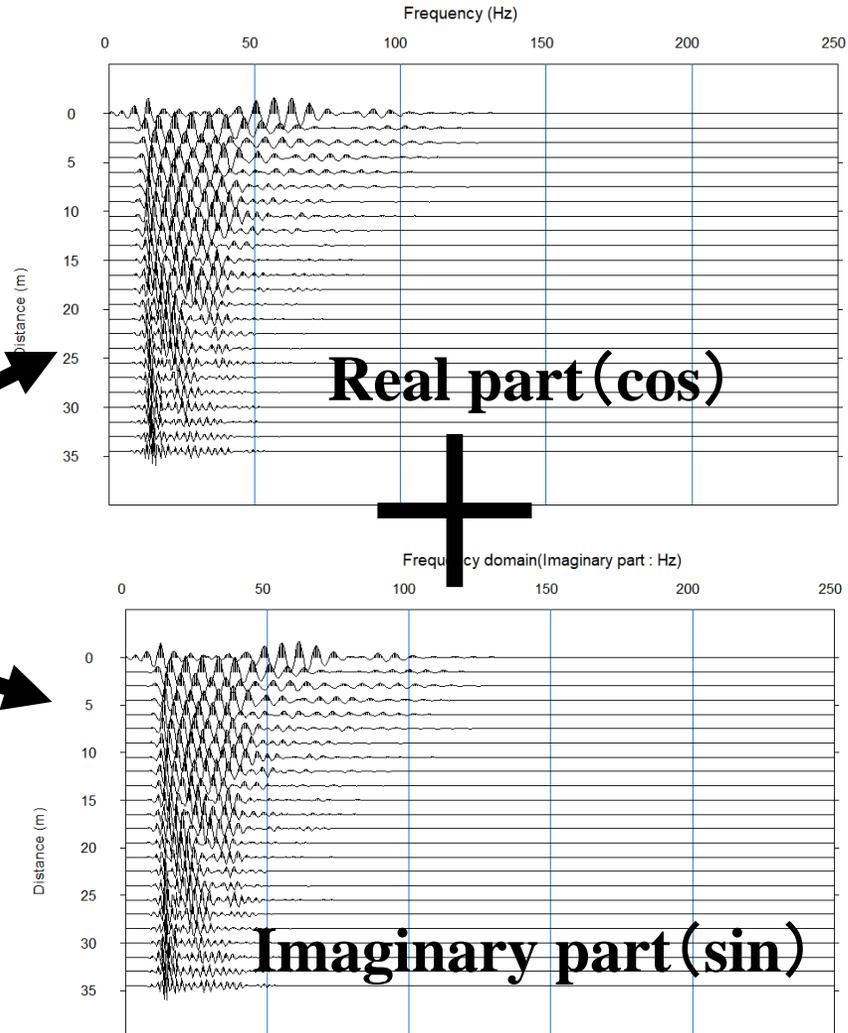
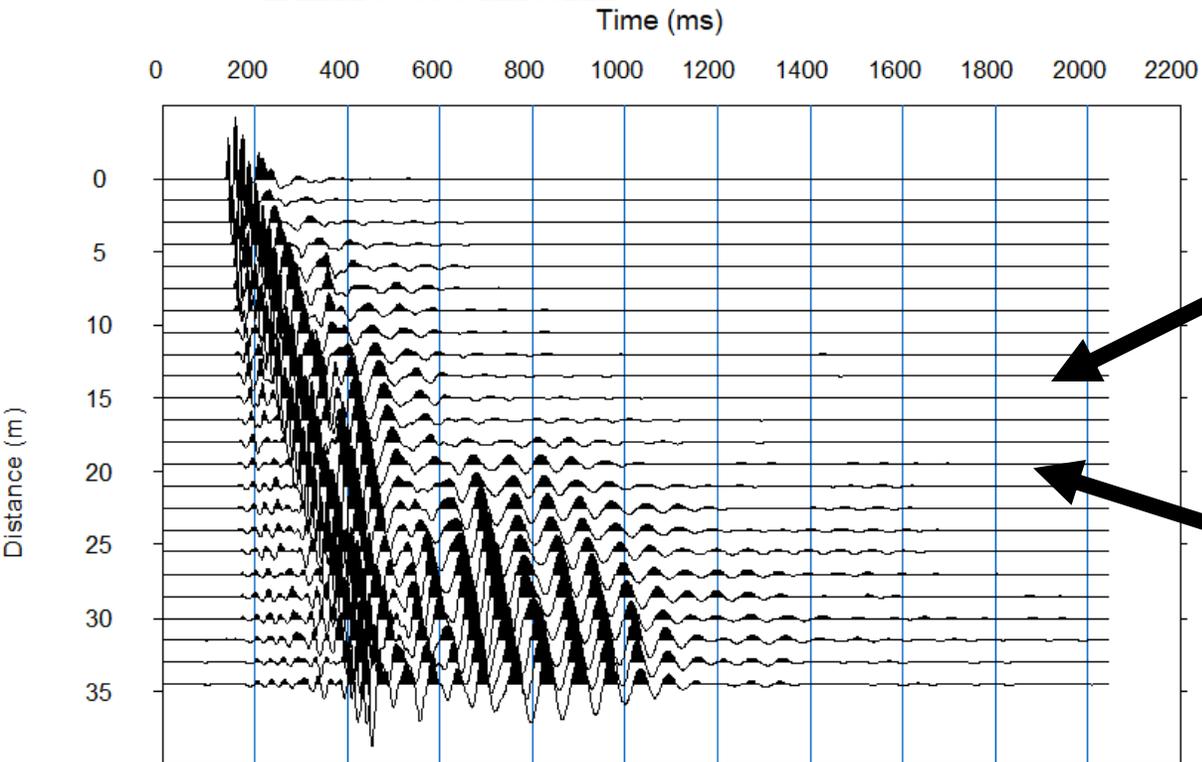
Time domain



Complex numbers

Frequency domain

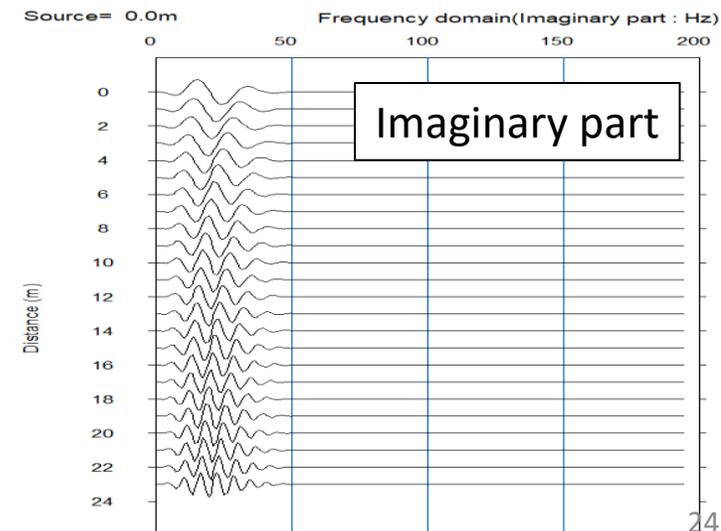
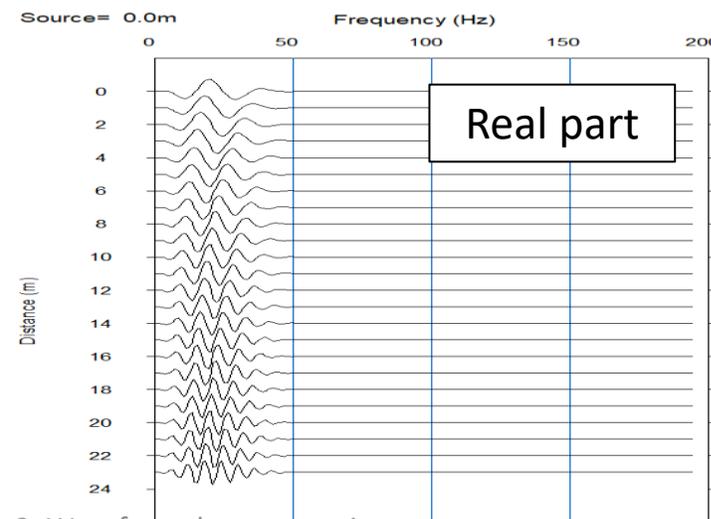
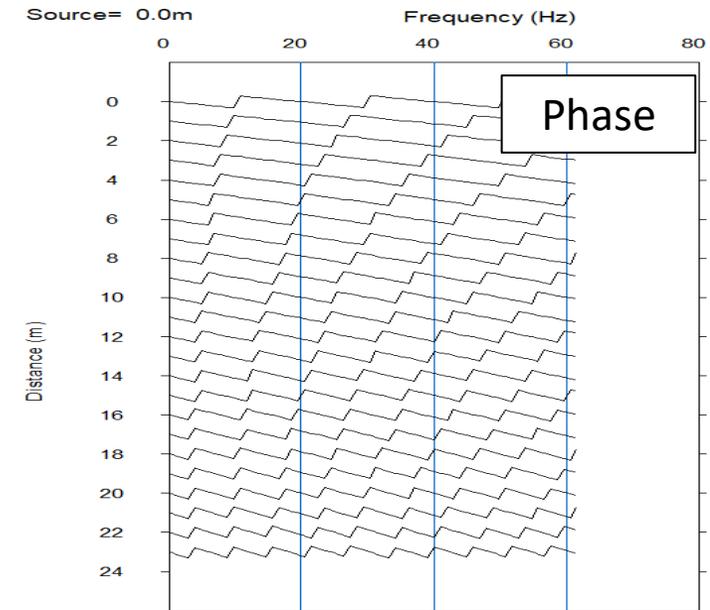
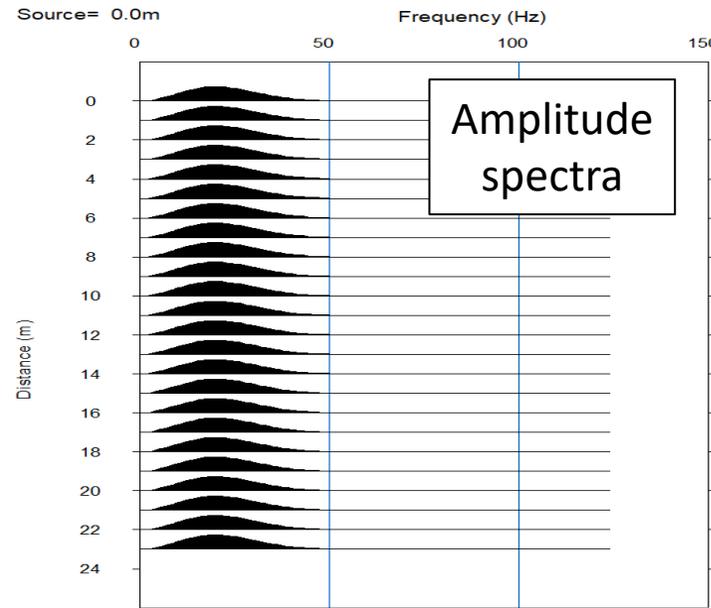
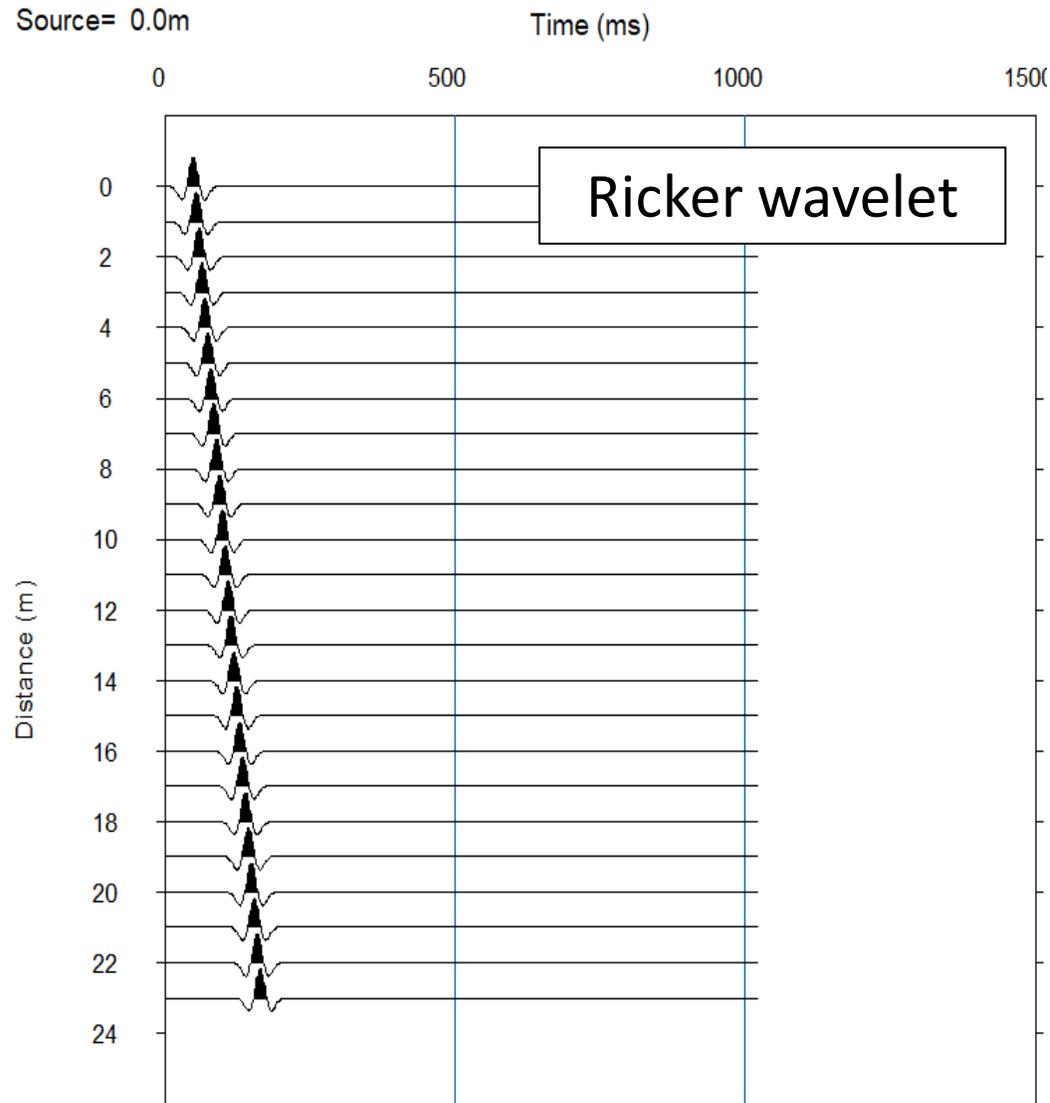
Time domain



Time domain, amplitude & phase, real & imaginary

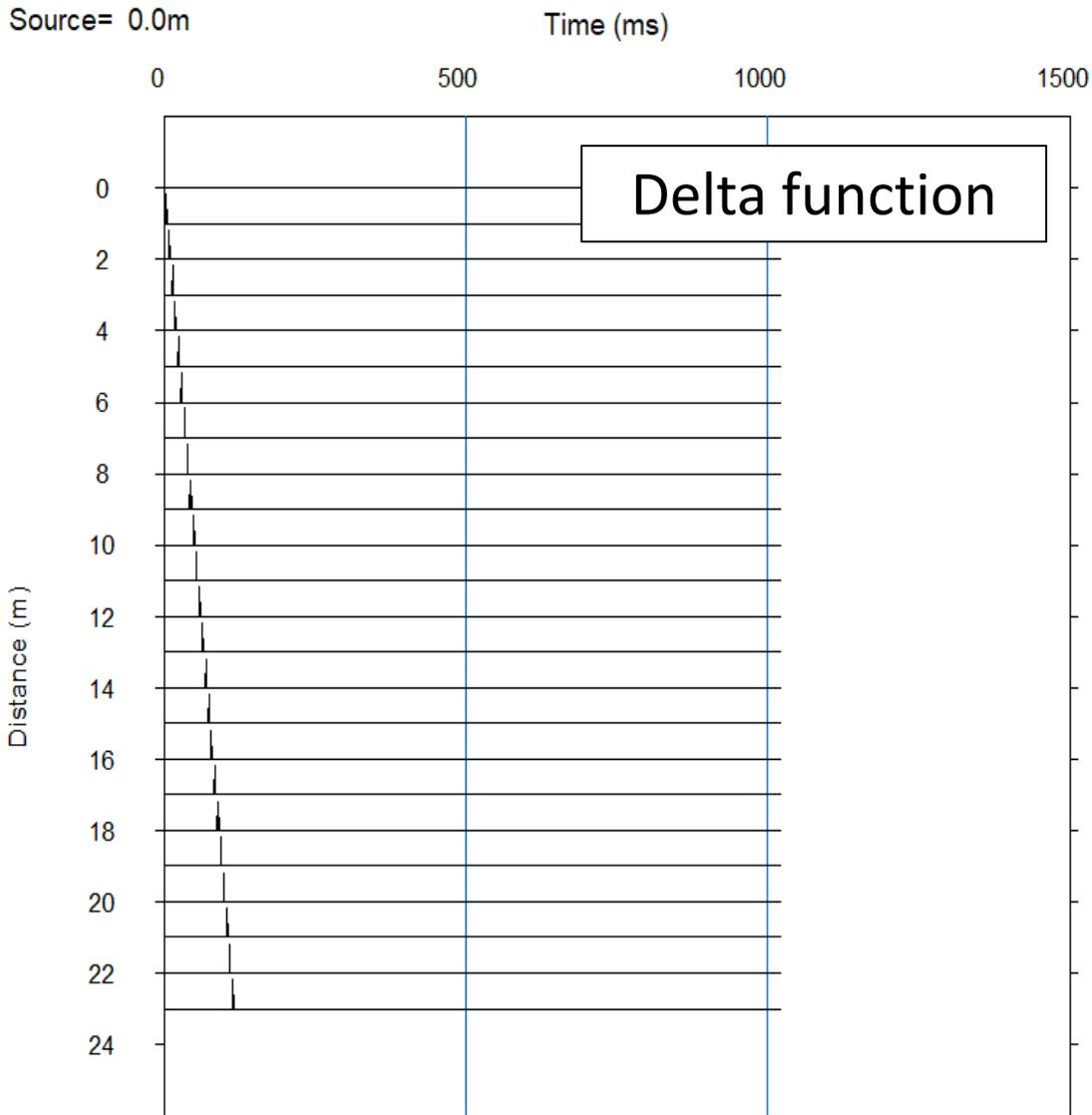
Time domain

Frequency domain

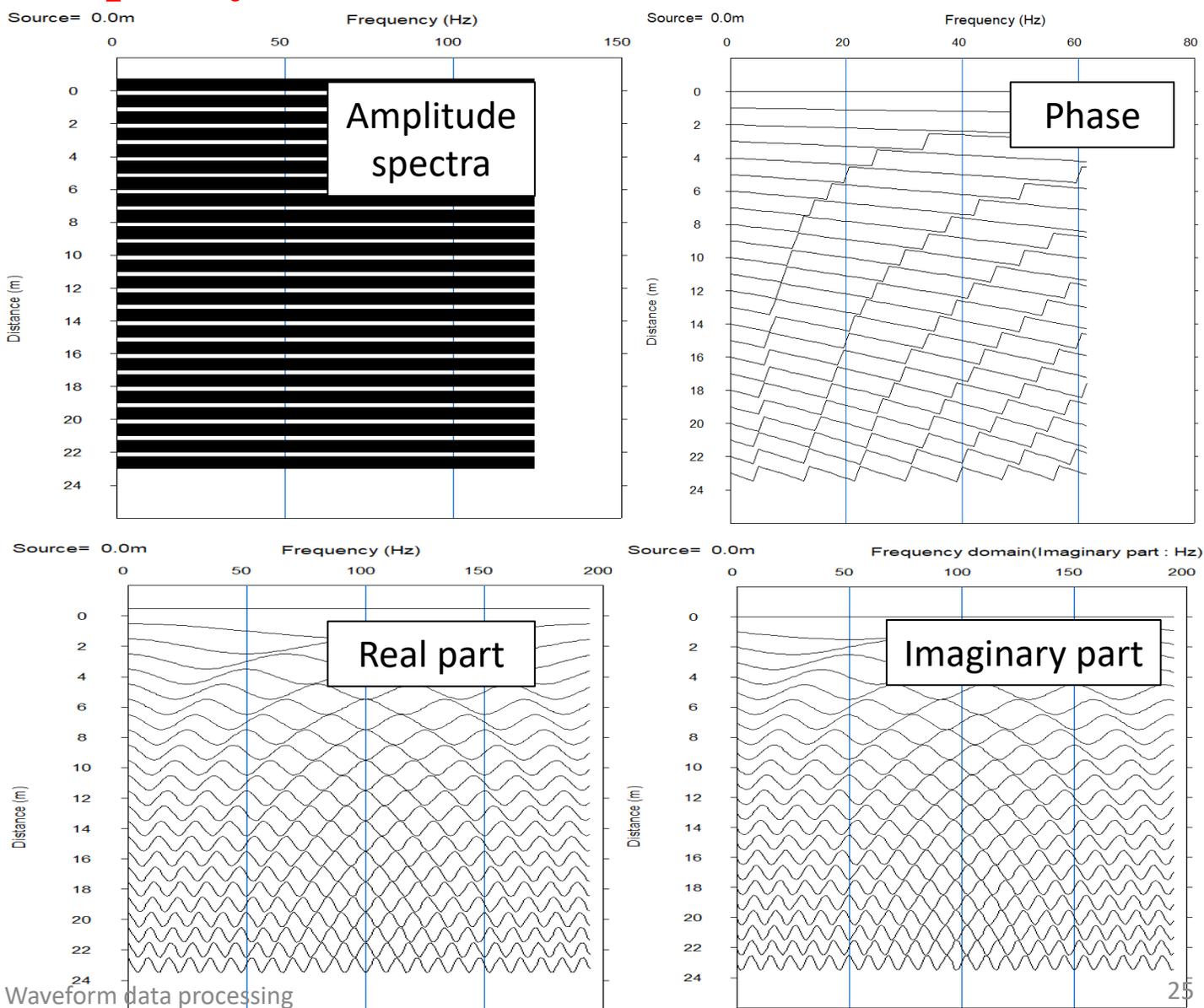


Time domain, amplitude & phase, real & imaginary

Time domain

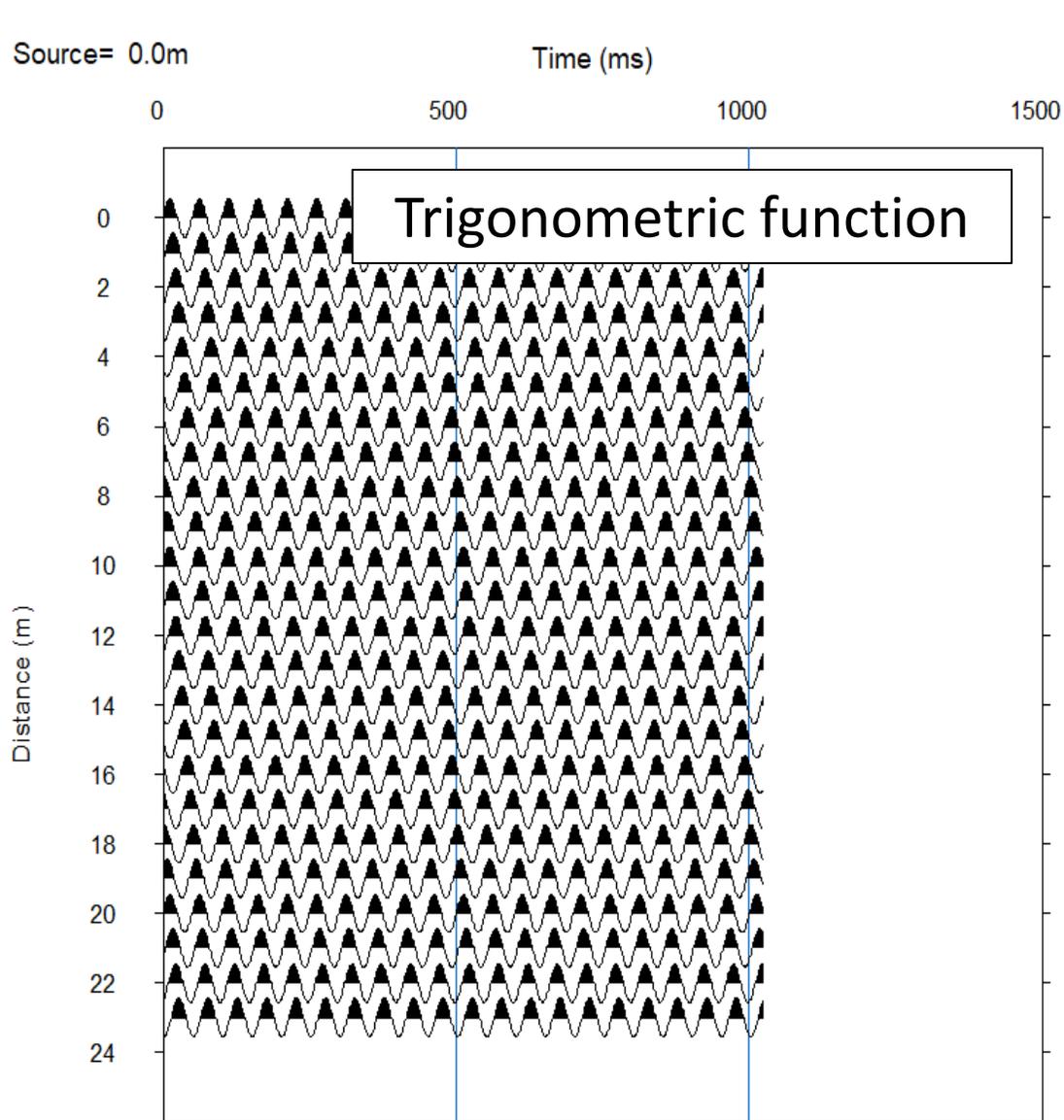


Frequency domain

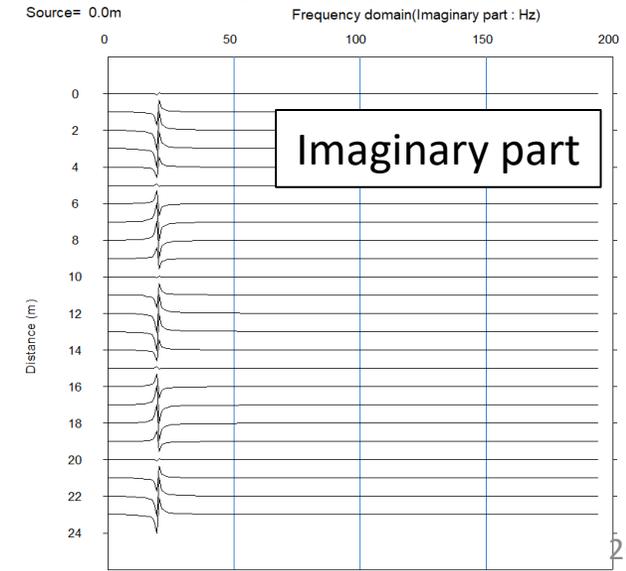
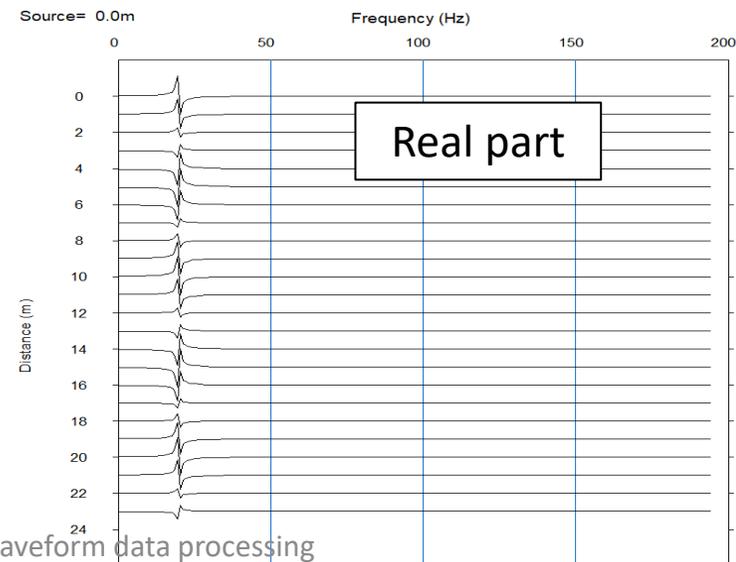
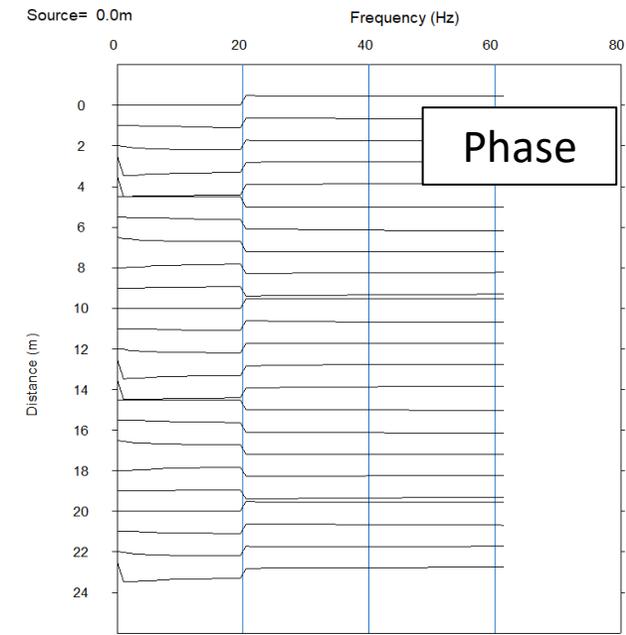
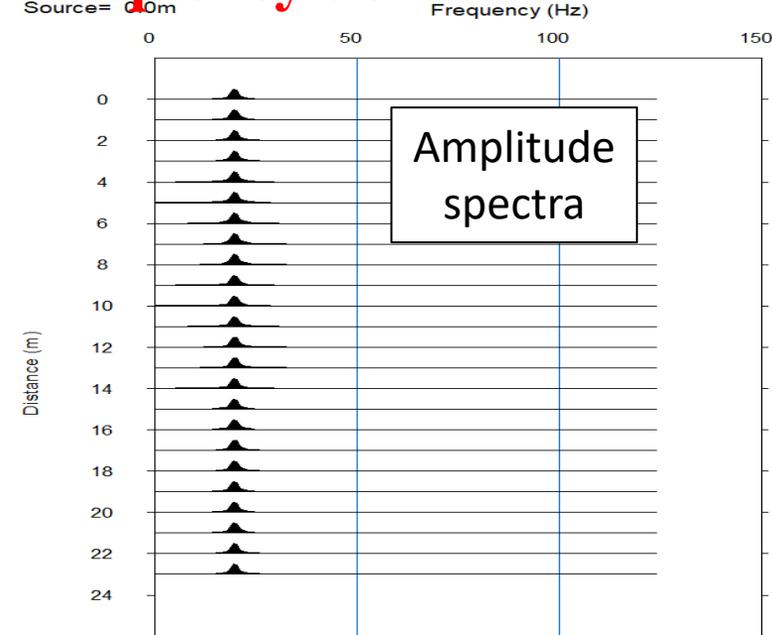


Time domain, amplitude & phase, real & imaginary

Time domain

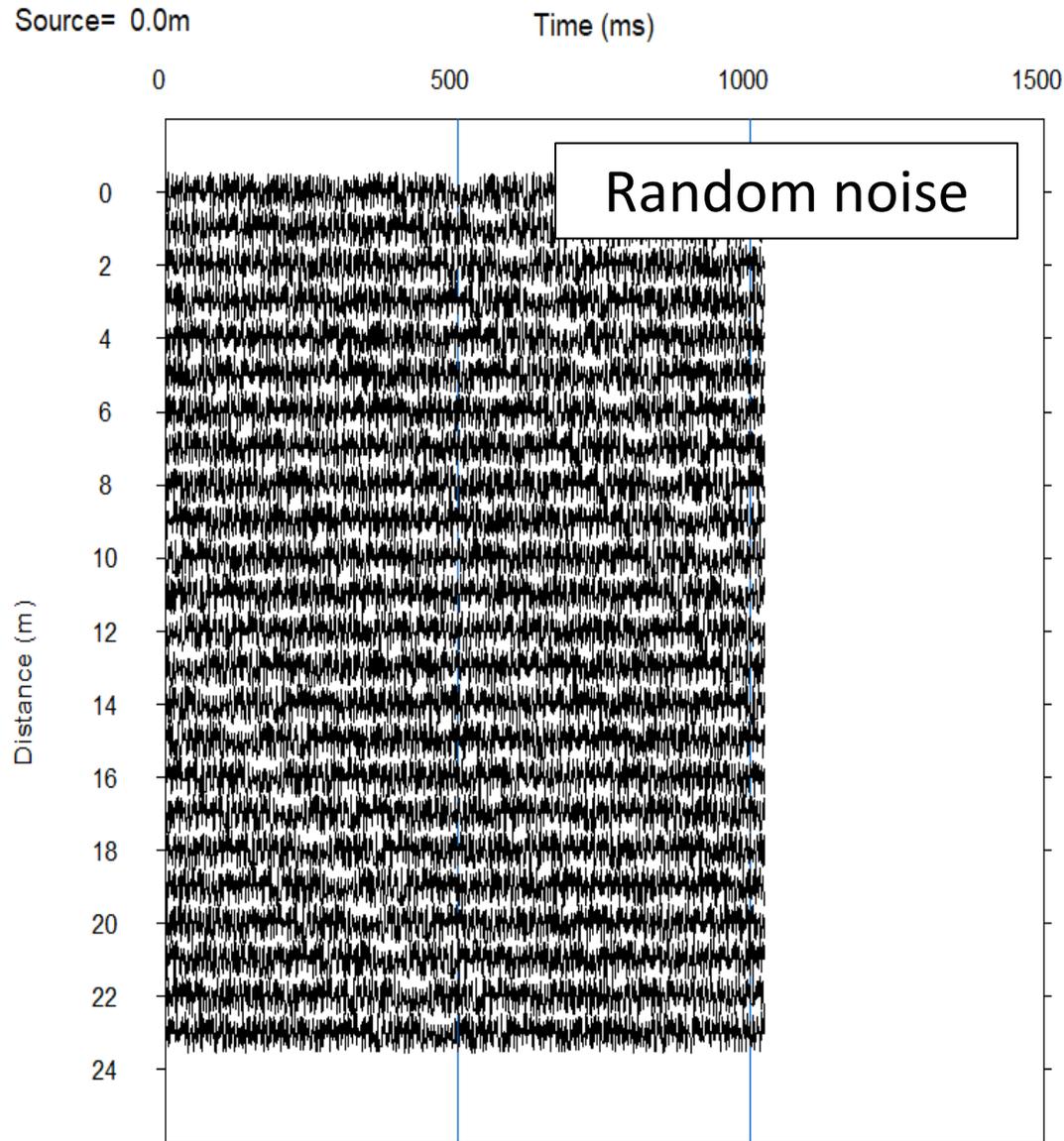


Frequency domain

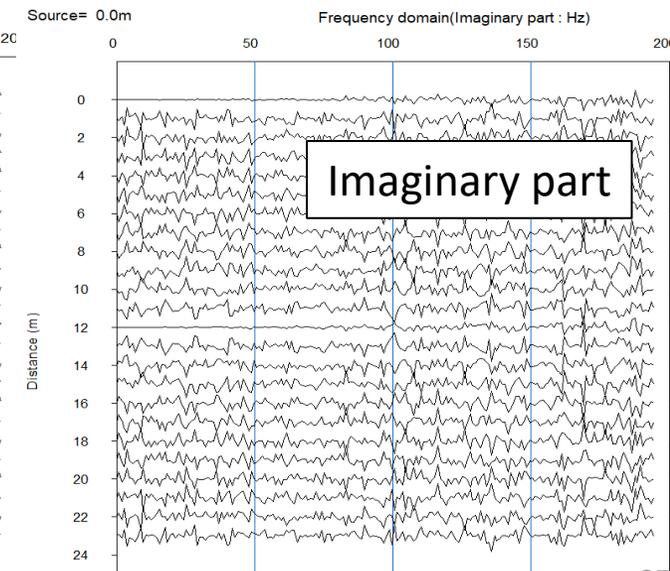
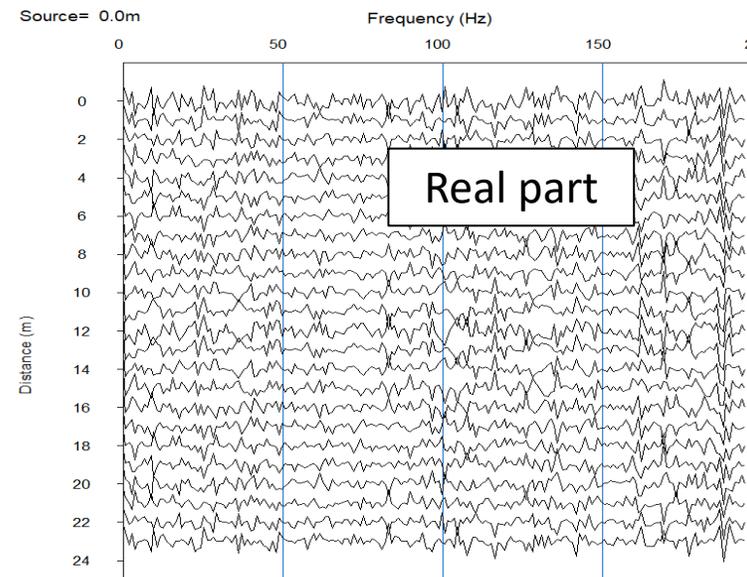
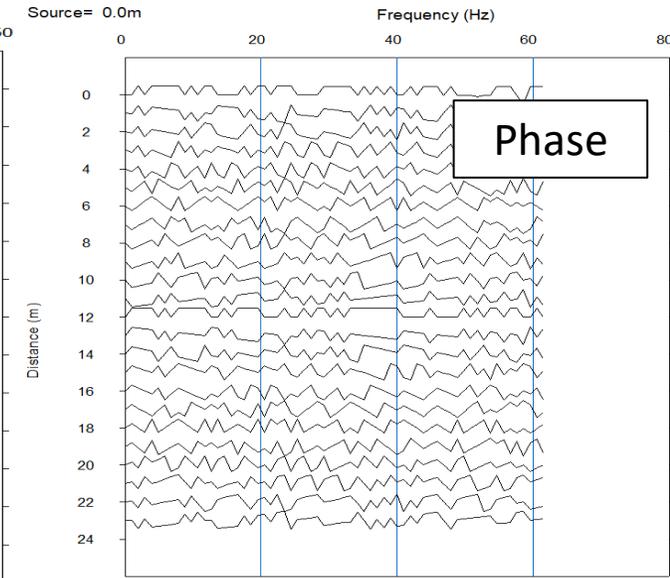
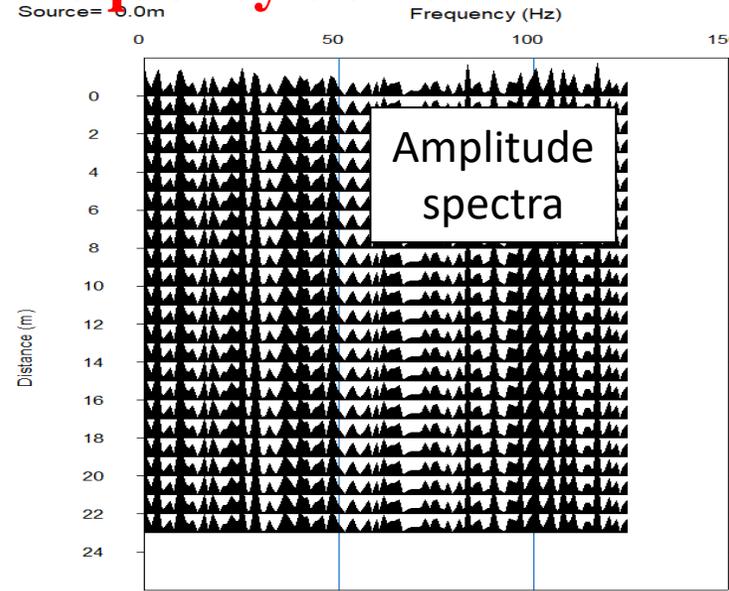


Time domain, amplitude & phase, real & imaginary

Time domain



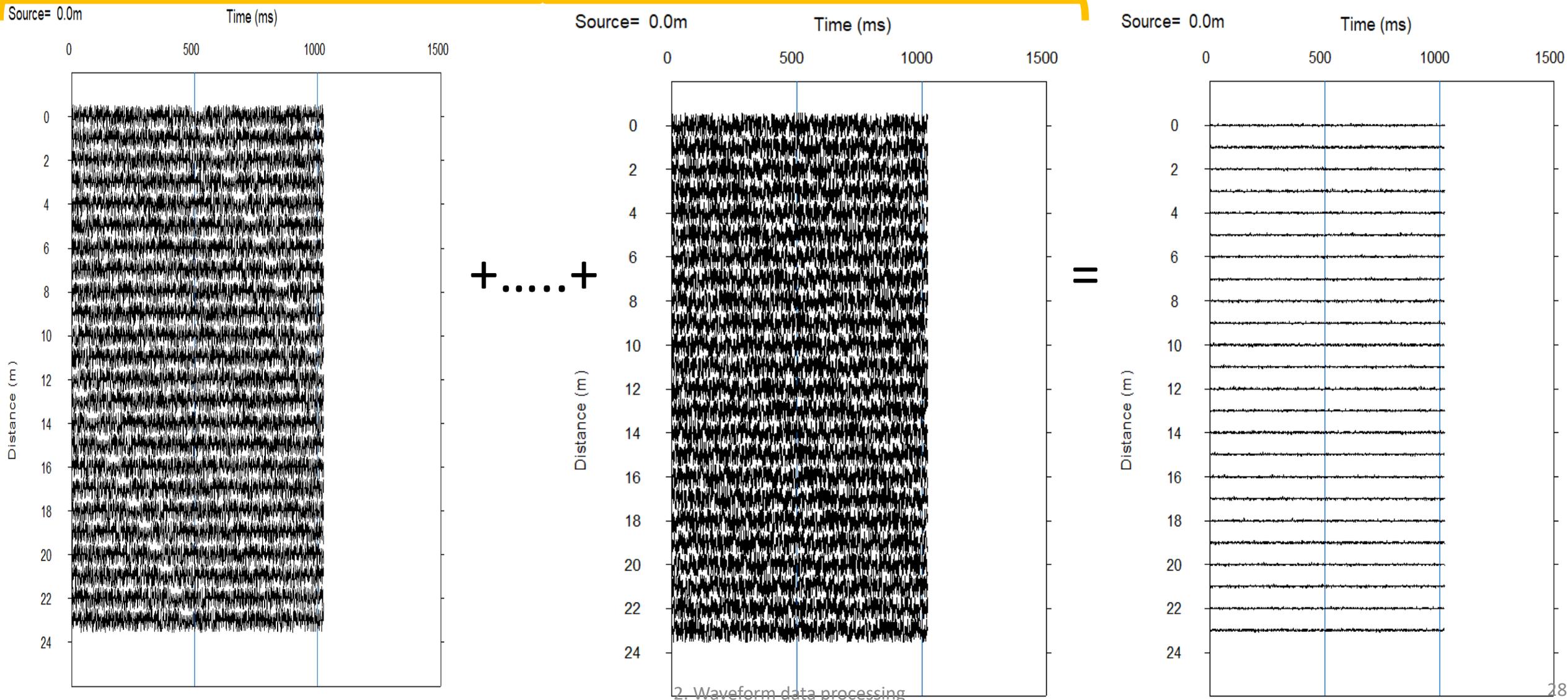
Frequency domain



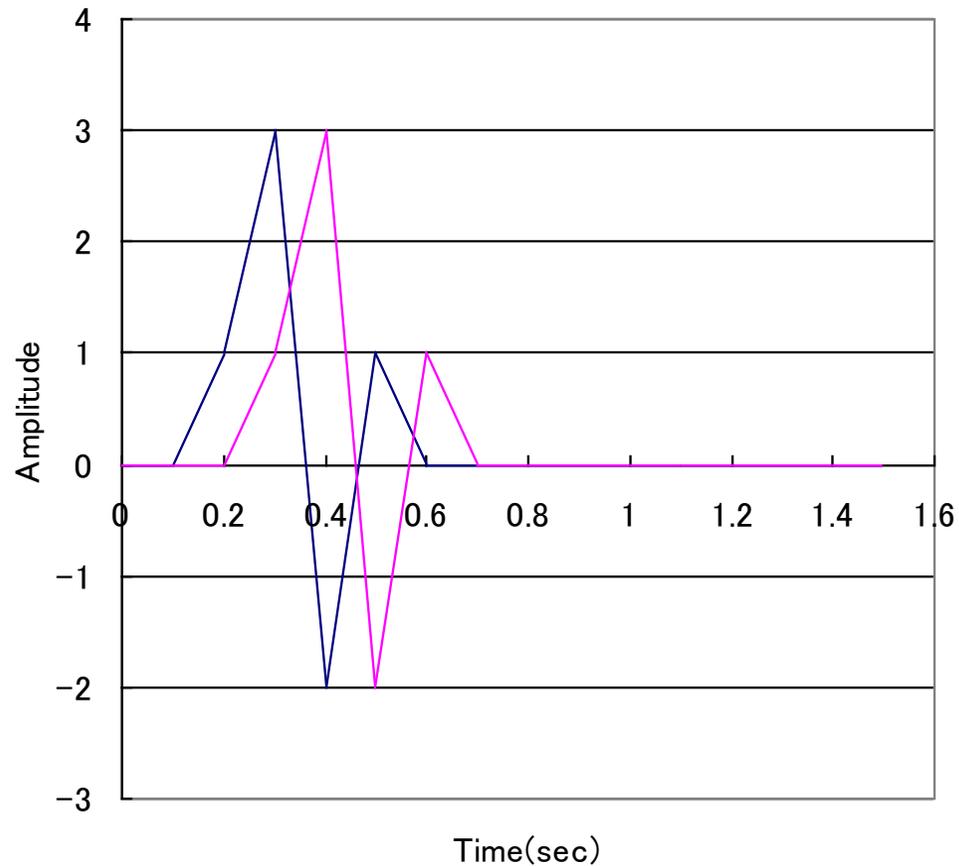
Ambient noise is random stochastic phenomena

Stack many files in time domain

Amplitude approaches to zero



Fourier transform using Excel (time domain data)



time(sec)	10m	20m
0	0	0
0.1	0	0
0.2	1	0
0.3	3	1
0.4	-2	3
0.5	1	-2
0.6	0	1
0.7	0	0
0.8	0	0
0.9	0	0
1	0	0
1.1	0	0
1.2	0	0
1.3	0	0
1.4	0	0
1.5	0	0

$$c = \frac{\Delta x}{\Delta t} = \frac{10}{0.1} = 100(m/s)$$

Fourier transform using Excel

Time domain			Frequency domain				
time(sec)	10m	20m	Frequency(Hz)	Re(10m)	Im(10m)	Re(20m)	Im(20m)
0	0	0	0	3	0	3	0
0.1	0	0	0.625	1.472474	-2.40262	0.440944	-2.78323
0.2	1	0	1.25	-0.82843	-2.41421	-2.29289	-1.12132
0.3	3	1	1.875	-2.55487	-1.17637	-2.06453	1.91021
0.4	-2	3	2.5	-3	2	2	3
0.5	1	-2	3.125	1.140652	4.237841	3.478745	-2.67558
0.6	0	1	3.75	4.828427	-0.41421	-3.70711	-3.12132
0.7	0	0	4.375	-0.05826	-4.98841	-1.85516	4.630986
0.8	0	0	5	-5	0	5	0
0.9	0	0	5.625	-0.05826	4.988411	-1.85516	-4.63099
1	0	0	6.25	4.828427	0.414214	-3.70711	3.12132
1.1	0	0	6.875	1.140652	-4.23784	3.478745	2.675577
1.2	0	0	7.5	-3	-2	2	-3
1.3	0	0	8.125	-2.55487	1.176373	-2.06453	-1.91021
1.4	0	0	8.75	-0.82843	2.414214	-2.29289	1.12132
1.5	0	0	9.375	1.472474	2.402625	0.440944	2.783227

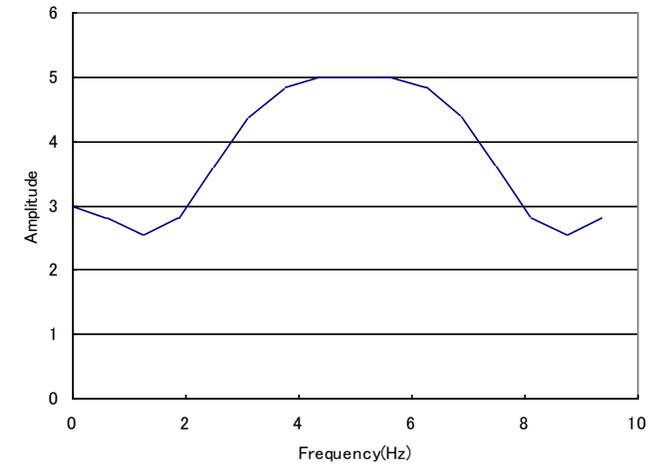


Fourier transform

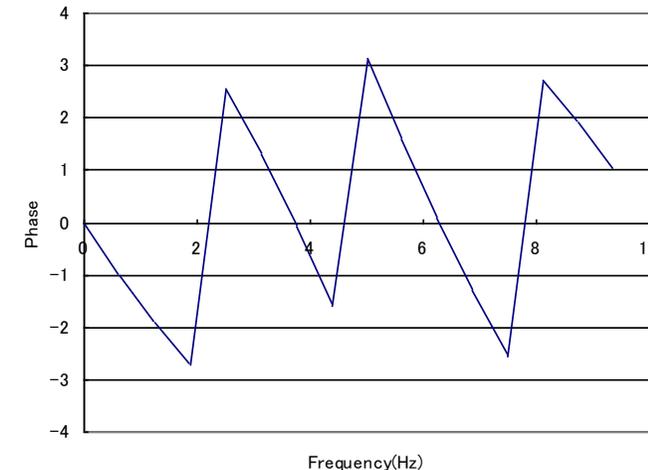
Fourier transform using Excel (Amplitude spectrum and phase spectrum)

Frequency(Hz)	Re(10m)	Im(10m)	Amplitude(10m)	Phase(10m)
0	3	0	3	0
0.625	1.472473646	-2.402624911	2.817939868	-1.020974282
1.25	-0.828427125	-2.414213562	2.552394685	-1.901351909
1.875	-2.554865846	-1.176373052	2.812684314	-2.710087387
2.5	-3	2	3.605551275	2.55359005
3.125	1.140652284	4.237840511	4.388664925	1.307868654
3.75	4.828427125	-0.414213562	4.846161509	-0.085576919
4.375	-0.058260084	-4.988411349	4.98875155	-1.582474882
5	-5	0	5	3.141592654
5.625	-0.058260084	4.988411349	4.98875155	1.582474882
6.25	4.828427125	0.414213562	4.846161509	0.085576919
6.875	1.140652284	-4.237840511	4.388664925	-1.307868654
7.5	-3	-2	3.605551275	-2.55359005
8.125	-2.554865846	1.176373052	2.812684314	2.710087387
8.75	-0.828427125	2.414213562	2.552394685	1.901351909
9.375	1.472473646	2.402624911	2.817939868	1.020974282

Amplitude spectrum



Phase spectrum



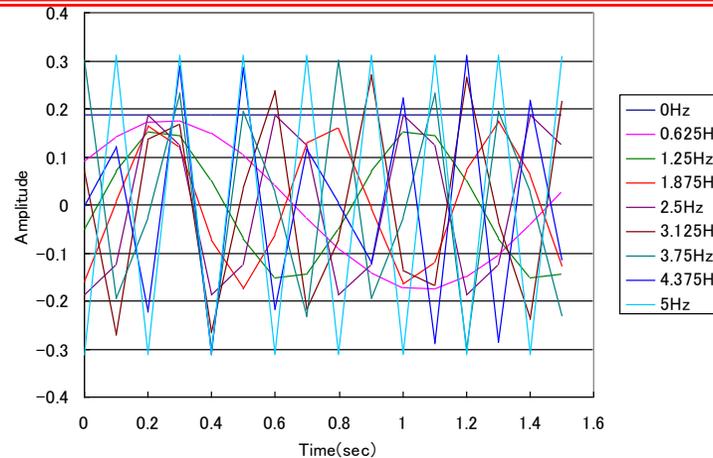
Fourier transform using Excel (Decomposition in Fourier series)

Frequency(Hz)	Amplitude(10m)	Phase(10m)	n	Time (sec)																
				0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1	1.1	1.2	1.3	1.4	1.5	
0	3	0	0	0.19	0.19	0.19	0.19	0.19	0.19	0.19	0.19	0.19	0.19	0.19	0.19	0.19	0.19	0.19	0.19	
0.625	2.817939868	-1.020974282	1	0.09	0.14	0.17	0.17	0.15	0.1	0.04	-0	-0.1	-0.1	-0.2	-0.2	-0.2	-0.1	-0	0.03	
1.25	2.552394685	-1.901351909	2	-0.1	0.07	0.15	0.14	0.05	-0.1	-0.2	-0.1	-0.1	0.07	0.15	0.14	0.05	-0.1	-0.2	-0.1	
1.875	2.812684314	-2.710087387	3	-0.2	0.01	0.16	0.12	-0.1	-0.2	-0.1	0.13	0.16	-0	-0.2	-0.1	0.07	0.18	0.06	-0.1	
2.5	3.605551275	2.55359005	4	-0.2	-0.1	0.19	0.13	-0.2	-0.1	0.19	0.13	-0.2	-0.1	0.19	0.13	-0.2	-0.1	0.19	0.13	
3.125	4.388664925	1.307868654	5	0.07	-0.3	0.14	0.17	-0.3	0.04	0.24	-0.2	-0.1	0.27	-0.1	-0.2	0.26	-0	-0.2	0.22	
3.75	4.846161509	-0.085576919	6	0.3	-0.2	-0	0.23	-0.3	0.2	0.03	-0.2	0.3	-0.2	-0	0.23	-0.3	0.2	0.03	-0.2	
4.375	4.98875155	-1.582474882	7	-0	0.12	-0.2	0.29	-0.3	0.29	-0.2	0.12	0	-0.1	0.22	-0.3	0.31	-0.3	0.22	-0.1	
5	5	3.141592654	8	-0.3	0.31	-0.3	0.31	-0.3	0.31	-0.3	0.31	-0.3	0.31	-0.3	0.31	-0.3	0.31	-0.3	0.31	
5.625	4.98875155	1.582474882	9	-0	0.12	-0.2	0.29	-0.3	0.29	-0.2	0.12	0	-0.1	0.22	-0.3	0.31	-0.3	0.22	-0.1	
6.25	4.846161509	0.085576919	10	0.3	-0.2	-0	0.23	-0.3	0.2	0.03	-0.2	0.3	-0.2	-0	0.23	-0.3	0.2	0.03	-0.2	
6.875	4.388664925	-1.307868654	11	0.07	-0.3	0.14	0.17	-0.3	0.04	0.24	-0.2	-0.1	0.27	-0.1	-0.2	0.26	-0	-0.2	0.22	
7.5	3.605551275	-2.55359005	12	-0.2	-0.1	0.19	0.13	-0.2	-0.1	0.19	0.12	-0.2	-0.1	0.19	0.12	-0.2	-0.1	0.19	0.12	
8.125	2.812684314	2.710087387	13	-0.2	0.01	0.16	0.12	-0.1	-0.2	-0.1	0.13	0.16	-0	-0.2	-0.1	0.07	0.18	0.06	-0.1	
8.75	2.552394685	1.901351909	14	-0.1	0.07	0.15	0.14	0.05	-0.1	-0.2	-0.1	-0.1	0.07	0.15	0.14	0.05	-0.1	-0.2	-0.1	
9.375	2.817939868	1.020974282	15	0.09	0.14	0.17	0.17	0.15	0.1	0.04	-0	-0.1	-0.1	-0.2	-0.2	-0.2	-0.1	-0	0.03	

$$a = A(f) \cdot \cos(\phi(f) + 2\pi n \cdot t/1.6)$$

$A(f)$: Amplitude
 $\phi(f)$: Phase

Total	0	0	1	3	-2	1	-0	-0	-0	-0	-0	-0	0	0	-0	0	0
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Fourier transform by Python

```
import numpy as np
import matplotlib.pyplot as plt
import math
```

```
f=[0,0,1,3,-2,1,0,0,0,0,0,0,0,0,0]
g=[0,0,0,1,3,-2,1,0,0,0,0,0,0,0,0]
```

```
plt.plot(f)
plt.plot(g)
```

```
F = np.fft.fft(f)
G = np.fft.fft(g)
```

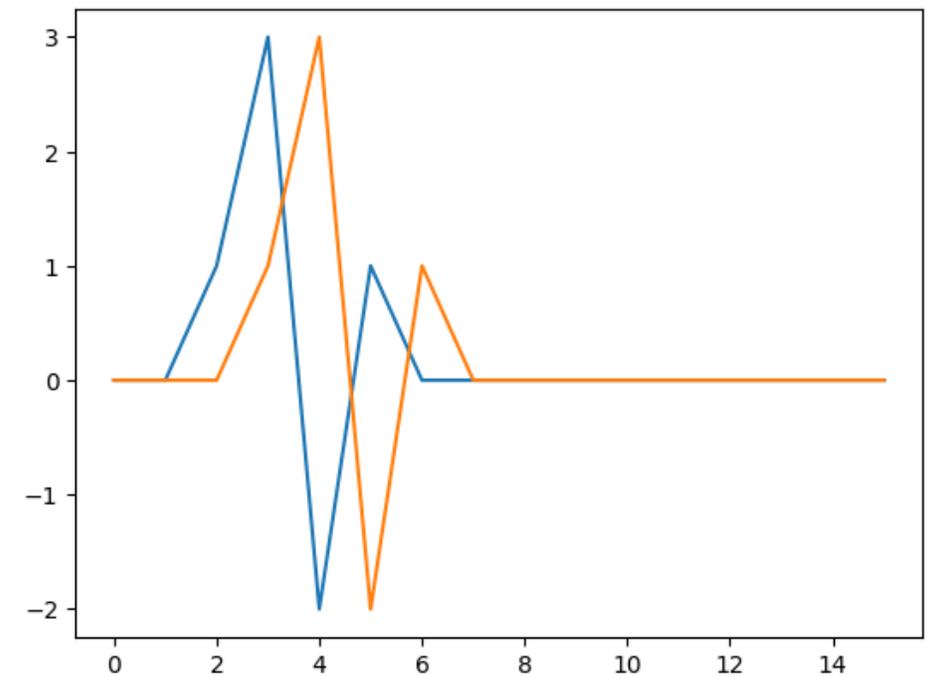
```
freq = np.fft.fftfreq(16, d=0.1)
```

```
print(F)
print(freq)
```

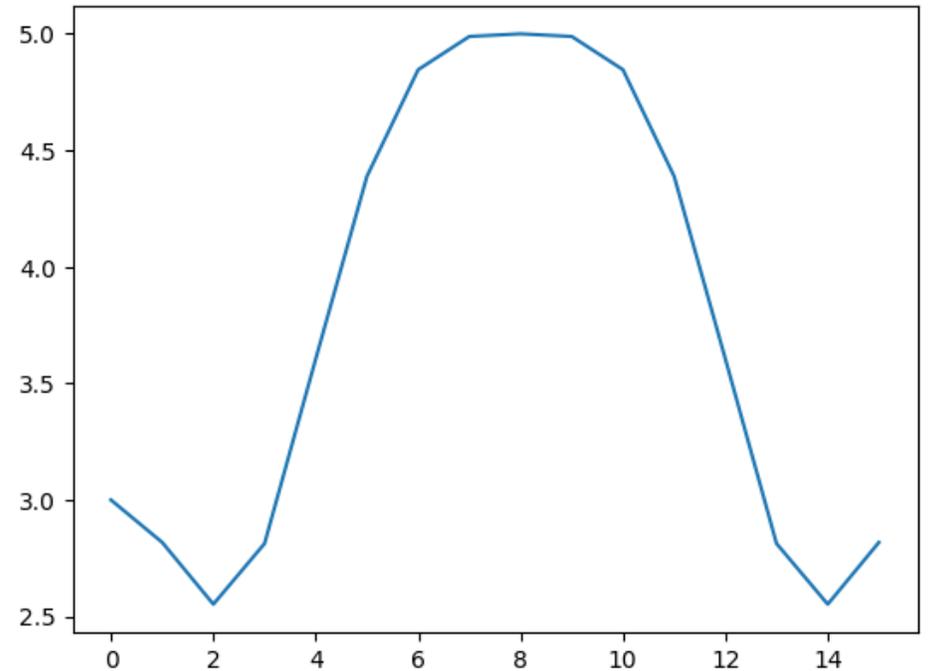
```
plt.plot(abs(F))
```

fft and cc.ipynb

➔ Results



➔ Results



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